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Harmonic Maps and Their Existence on Complete Riemannian Manifolds

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Abstract:

Harmonic Maps and Their Existence on Complete Riemannian Manifolds Abstract The theory of harmonic maps, initiated by Eells and Sampson in 1964, studies critical points of the Dirichlet energy functional for maps between Riemannian manifolds. When the domain is compact and the target has non-positive sectional curvature, the harmonic map heat flow converges to a smooth harmonic representative in every homotopy class. The situation becomes substantially more delicate when the domain is a complete non-compact Riemannian manifold. In this setting, existence is no longer guaranteed by compactness and requires additional analytic and geometric hypotheses: lower bounds on Ricci curvature, the existence of a positive Green's function, finite energy or controlled growth of the tension field, and curvature conditions on the target.

This paper surveys the principal existence theorems for harmonic maps from complete Riemannian manifolds. We recall the classical Eells–Sampson heat-flow method and its extensions by Schoen–Yau, Chen–Li, Li–Tam and others. Particular attention is paid to the role of the Bochner formula, gradient estimates, and the interplay between the spectrum of the Laplace–Beltrami operator and the existence of finite-energy harmonic maps. Liouville-type rigidity results, which assert that finite-energy harmonic maps from manifolds with non-negative Ricci curvature into non-positively curved targets must be constant, are also discussed as complementary non-existence statements. The survey concludes with open problems concerning existence under weaker curvature assumptions and the behaviour of the heat flow at infinity.

Keywords: Harmonic maps; Complete Riemannian manifolds; Dirichlet energy; Tension field and Non-positive curvature etc.

Introduction:

A smooth map $f : M, g \to N, h$ between Riemannian manifolds is called harmonic if it is a critical point of the Dirichlet energy

$$E(f)\frac{1}{2}_M |df|^2 \, dvol_g$$

The Euler–Lagrange equation associated with this functional is the vanishing of the tension field

$$\tau(f) = tr_g \nabla df = 0$$

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In local coordinates this is a semi-linear elliptic system whose leading term is the Laplace–Beltrami operator of *M*. When *M* is compact without boundary and the sectional curvature of *N* is non-positive, the celebrated theorem of Eells and Sampson 1964 asserts that every continuous map is homotopic to a smooth harmonic map. The proof proceeds by solving the harmonic map heat flow

$$\partial_t f = \tau\left(f_t\right),\ f_0 = f$$

and establishing long-time existence together with smooth convergence as $t \to \infty$. The non-positivity of the curvature of N supplies the necessary a-priori estimates via the Bochner identity

$$\frac{1}{2}\Delta\left|df\right|^2 = \left|\nabla df\right|^2 + \left\langle Ric^M, df \otimes df \right\rangle - \left\langle R^N, (df, df)df, df \right\rangle$$

The last term is non-negative when operatorname $\sec^N \le 0$, and the resulting differential inequality permits the application of the maximum principle.

Once the domain ceases to be compact the situation changes dramatically. Completeness of *M* alone does not guarantee the existence of a harmonic map in a given homotopy class, nor even the existence of a finite-energy harmonic map with prescribed asymptotic behaviour. The fundamental difficulties are the possible non-existence of a positive Green's function, the lack of a uniform spectral gap, and the possible escape of energy to infinity under the heat flow. Consequently, additional geometric hypotheses must be imposed.

Schoen and Yau 1976, 1979 obtained the first major results in the complete setting. They proved that if *M* is complete with non-negative Ricci curvature and *N* is compact with non-positive sectional curvature, then every harmonic map of finite energy is constant. The argument relies on the Bochner formula together with Yau's theorem that non-negative L^2-subharmonic functions on complete manifolds of non-negative Ricci curvature are constant. This Liouville-type theorem has far-reaching topological consequences: it implies, for example, that a complete manifold of non-negative Ricci curvature cannot admit a non-constant finite-energy map into a compact manifold of non-positive curvature, and therefore imposes strong restrictions on the fundamental group.

Existence theorems on complete manifolds typically require a combination of curvature bounds, volume-growth conditions and integrability assumptions on the initial map. Chen and Li 1999 studied the heat flow when the Ricci curvature of *M* is bounded from below by a negative constant, M admits a positive Green's function, and a certain integral involving the tension field of the initial map remains controlled. Under these hypotheses they established uniform convergence of the heat flow and hence the existence of a harmonic map. Li and Tam, and later extensions by Donnelly and others, obtained existence when *M* has bounded geometry and a positive lower bound on the spectrum of the Laplacian, while the target is a complete simply-connected manifold of non-positive curvature. In this framework every C^2 map whose tension field lies in L^p for some $p \in 1, \infty$ is at a bounded distance from a harmonic map.

A different line of research concerns maps of finite energy. Schoen and Yau showed that a finite-energy map from a complete manifold into a compact non-positively curved manifold is homotopic on compact sets to a harmonic map of no greater energy. The construction again uses a heat-flow argument, but now the finite-energy assumption replaces compactness and guarantees that the flow remains in a weakly compact subset of the Sobolev space $W^{1,2}$.

More recent work has relaxed the curvature assumptions on the target or the domain. Results of the form "every map of controlled growth is homotopic to a harmonic map" have been obtained when the target is a Hadamard manifold complete, simply connected, non-positive curvature and the domain satisfies a volume-doubling condition together with a Poincaré inequality. Parallel developments exist for Hermitian harmonic maps, p-harmonic maps and maps into singular NPC spaces Alexandrov spaces of non-positive curvature. The latter theory, initiated by Gromov–Schoen and Korevaar–Schoen, replaces the classical tension-field equation by a variational inequality and yields existence of energy-minimizing maps into CAT(0) spaces under very mild hypotheses on the domain.
Despite these advances, several fundamental questions remain open. It is still not known whether a complete manifold of positive Ricci curvature admits a non-constant harmonic map into a compact manifold of non-positive curvature when the energy is allowed to be infinite. The behaviour of the harmonic map heat flow at spatial infinity on manifolds with slow volume growth is only partially understood. The relationship between the existence of harmonic maps and the spectrum of the Laplacian continues to be an active area of investigation.

Analytic Preliminaries and Existence Theorems

Let (M, g) be a complete Riemannian manifold of dimension m and (N, h) a complete Riemannian manifold of dimension n. The energy density of a map $f : M \to N$ is $e(f) = \frac{1}{2}|df|^2$. A map is harmonic if and only if its tension field vanishes identically. The Bochner formula for the energy density of a harmonic map reads

$$\Delta e(f) = |\nabla df|^2 + Ric^M(\nabla f, \nabla f) - \langle R^N(df, df)df, df \rangle$$

when $\sec^N \leq 0$ the curvature term is non-negative, and one obtains the differential inequality

$$\Delta e(f) \geq |\nabla df|^2 + Ric^M(\nabla f, \nabla f)$$

If in addition $Ric^M \geq 0$, then $e(f)$ is subharmonic. On a complete manifold this implies, via the maximum principle or Yau's L^2-Liouville theorem, strong rigidity statements.
Existence via the heat flow on complete manifolds proceeds as follows. One solves the parabolic system

$$\partial_t f = \tau(f_t)$$

with initial data f_0 of finite energy or of controlled growth. Short-time existence follows from standard parabolic theory once the initial map is sufficiently regular. Long-time existence requires a-priori estimates that prevent the energy density from blowing up in finite time. When $\sec^N \leq 0$ the Bochner inequality yields

$$\partial_t e(f) \leq \Delta e(f) + Ce(f)$$

where the constant C depends on a lower bound for Ric^M. If M admits a positive Green's function $G(x, y)$, one can represent the solution by means of the heat kernel and obtain integral bounds of the form

$${}_M G(x,y)\,|\tau(f_0)(y)|\,dvol_y < \infty$$

on compact sets. Under this hypothesis Chen and Li proved that the heat flow exists for all time and converges uniformly on compact subsets to a harmonic map.

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When M has bounded geometry Ricci curvature bounded below and positive injectivity radius and a spectral gap $\lambda_1(M) > 0$, Li–Tam-type arguments show that every map whose tension field belongs to $L^p(M)$ for some $p\in(1, \infty)$ is at a bounded distance from a harmonic map into a Hadamard manifold. The spectral gap supplies exponential decay of the heat kernel, which compensates for the lack of compactness.

For finite-energy maps the situation is simpler. Let $E(f_0)<\infty$ and $\sec^N \leq 0$. The energy is non-increasing along the heat flow, and the L^2-bound on df_t together with the Bochner inequality produce uniform local $C^{1,\alpha}$ estimates. Passing to the limit yields a harmonic map of energy at most $E(f_0)$ that is homotopic to f_0 on every compact subset of M.

Liouville theorems complement the existence theory. The classical Schoen–Yau result states that a finite-energy harmonic map from a complete manifold of non-negative Ricci curvature into a compact manifold of non-positive sectional curvature is constant. The proof combines the Bochner formula with the fact that a non-negative L^2-subharmonic function on such a manifold must be constant. Variants exist under weaker curvature assumptions non-negative Bakry–Émery Ricci curvature, quasi-positive Ricci curvature and for p-harmonic maps.

Conclusion:

The existence theory of harmonic maps on complete Riemannian manifolds rests on a delicate balance between curvature conditions, analytic estimates and integrability assumptions. When the target has non-positive sectional curvature, the heat-flow method, suitably adapted by means of Green's functions, spectral gaps or finite-energy hypotheses, produces harmonic maps under a variety of geometric hypotheses on the domain. Complementary Liouville theorems demonstrate that finite-energy harmonic maps are frequently rigid, often forcing constancy when the domain has non-negative Ricci curvature.

Despite substantial progress since the foundational work of Eells–Sampson and Schoen–Yau, several questions remain open. The existence of infinite-energy harmonic maps on manifolds of positive Ricci curvature, the long-time behaviour of the heat flow on manifolds with slow volume growth, and the extension of the theory to targets of mixed curvature are among the most pressing problems. Further investigation of the interplay between the spectrum of the Laplacian and the existence of harmonic maps is likely to yield new insights. The theory continues to serve as a powerful tool in geometric analysis, topology and the study of manifolds of non-negative curvature.

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